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On an Unconditionally Stable Difference Scheme for the Multifrequency Eddington Equations*

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Various numerical procedures have been used for the direct discretization of the relevant equations governing the flow of radiation through materials which scatter, emit, and absorb that radiation. The solution of certain problems has in practice required procedures which are unconditionally stable; such procedures have been developed for which numerical tests have indicated unconditional stability for problems of interest. Generally, however, proofs of unconditional stability have been given only for grey or one-group problems with no account being taken of the coupling over different frequency groups. In the present paper we give a stability analysis for the Eddington approach which takes into account this coupling over different frequency groups. We consider difference equations similar to the corresponding equations used successfully in Ref. [1]. These equations are not fully implit since it was found necessary to include certain terms in the difference equations in an explicit manner in order to avoid prohibitively long calculation times for multifrequency initial-boundary value problems (cf. Ref. [1]). On the other hand it was crucial in Ref. [1] that the equations be unconditionally stable. We prove here that the given scheme is indeed unconditionally stable for the initial value problem with multifrequency scattering, emission, and absorption of radiation. Our proof is direct and is based on an estimate for a certain energy norm of the solution.

1. INTRODUCTION

The equation of radiative transfer governing the flow of radiation through materials assumed to be in local thermodynamic equilibrium can be given quite generally as

$$\left(\frac{1}{c}\frac{\partial}{\partial t} + \boldsymbol{\Omega}\cdot\nabla + \mu\right) I = \mu B + S, \tag{1.1}$$

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where $I = I(\Omega, \nu, \mathbf{r}, t)$ is the specific intensity of radiation with frequency ν at position $\mathbf{r}$ in direction Ω at time t, and the term $S = S(I; \Omega, \nu, \mathbf{r}, t)$ accounts for the scattering processes (cf. Ref. [2]). The term μ is the absorption coefficient corrected for stimulated emission and B is the emissive source function; both are properties of the material through which the radiation flows and depend also on the frequency ν, and both are determined functionally by the assumption of local thermodynamic equilibrium. For example, the source function $B = B(\nu, \theta)$ is given in terms of the material temperature θ by the well-known Planck law

$$B(\nu, \theta) = \frac{2h\nu^3}{c^2} (e^{h\nu/\theta} - 1)^{-1}. \tag{1.2}$$

The material temperature θ appearing in (1.2) is itself a functional of the radiation field including all frequencies. Specifically, letting ϵ_m denote the internal energy density of the material, and neglecting hydrodynamics for the purpose of this analysis,[1] then the internal energy equation can be given as (cf. Refs. [1, 3])

$$\frac{\partial \epsilon_m}{\partial t} = -\int_0^\infty c\mu \left[\frac{4\pi}{c} B(\nu, \theta) - E\right] d\nu, \tag{1.3}$$

where E denotes the radiation energy density defined as

$$E = E(\nu, \mathbf{r}, t) = \frac{1}{c} \int I(\Omega, \nu, \mathbf{r}, t)\, d\Omega, \tag{1.4}$$

with the latter integration being taken over all directions (i.e., Ω ranges over the surface of the unit sphere). Assuming, as is customary, that ϵ_m is a known function of θ,

$$\epsilon_m = \epsilon_m(\theta),$$

and using (1.2), the internal energy equation (1.3) can then be rewritten as

$$\epsilon_m'(\theta) \frac{\partial \theta}{\partial t} = -\int_0^\infty c\mu \left[\theta^3 \frac{4\pi}{c} B\left(\frac{\nu}{\theta}, 1\right) - E\right] d\nu. \tag{1.5}$$

The quantity $\epsilon_m'(\theta) = d\epsilon_m/d\theta$ is the specific heat of the material.

Finally, then, it is seen that Eqs. (1.1), (1.2), (1.4), and (1.5) yield a highly coupled system of equations involving all frequencies $0 \leqslant \nu < \infty$. It is not surprising that exact solutions have been found analytically only in very special cases. Consequently numerical procedures have been developed for use in solving various

[1] The material is assumed to be of unit density, and at rest.

classes of problems of interest (see Refs. [1, 3–5] for methods involving direct discretizations of the relevant equations).

Considerations of stability are exceedingly important for such numerical procedures, and have been given, for example, in Refs. [5–9] where, however, no account has been taken of the coupling over different frequency groups.[2] We are concerned in the present paper with a stability analysis for the variable Eddington approach which takes into account this frequency coupling.

2. The Variable Eddington Equations

Allowing for conservative scattering, the first two moments of Eq. (1.1) can be given as (cf. Ref. [2])

$$\begin{aligned} \frac{\partial E}{\partial t} + \nabla \cdot \mathbf{F} &= c\mu \left(\frac{4\pi}{c} B - E \right) \\ \frac{1}{c} \frac{\partial \mathbf{F}}{\partial t} + c\nabla \cdot \bar{P} &= -(\mu + \sigma)\mathbf{F}, \end{aligned} \tag{2.1}$$

where $\sigma = \sigma(\nu, \mathbf{r}, t)$ is the scattering cross section and where E, $\mathbf{F}$, and $\bar{P}$ are the radiation energy density, flux vector, and pressure tensor respectively, defined as (see (1.4))

$$E = E(\nu, \mathbf{r}, t) = \frac{1}{c} \int I(\boldsymbol{\Omega}, \nu, \mathbf{r}, t)\, d\Omega,$$

$$\mathbf{F} = \mathbf{F}(\nu, \mathbf{r}, t) = \int I(\cdot)\, \boldsymbol{\Omega}\, d\Omega,$$

and

$$\bar{P} = \bar{P}(\nu, \mathbf{r}, t) = \frac{1}{c} \int I(\cdot)\, \boldsymbol{\Omega}\boldsymbol{\Omega}\, d\Omega.$$

The crucial assumption in the variable Eddington approach involves the method of terminating the sequence of moment equations. For one dimensional plane or spherical geometry the variable Eddington factor $f = f(\nu, r, t)$ is defined by the relation

$$P = fE, \tag{2.2}$$

[2] A reviewer has called to our attention Ref. [14], where unconditional stability is proved in a certain sense (though not in our sense) for a discretization of the multigroup equations of reactor point kinetics for one delayed neutron group. Our methods differ from those of Ref. [14].

where P denotes either P_{xx} or P_{rr} in either plane or spherical geometry. Although f is defined by (2.2), the important point is that it may in practice be evaluated by independent analysis. Eddington [10] first used (2.2) with $f = \frac{1}{3}$ as an approximation in studying the interior of a star, while approaches for the evaluation of the variable Eddington factor in more general situations may be found in Refs. [1, 11, 12].

Using (2.2) we may rewrite (2.1) in plane geometry as

$$\begin{aligned} \frac{\partial E}{\partial t} + \frac{\partial F}{\partial x} &= c\mu \left(\frac{4\pi}{c} B - E \right), \\ \frac{\partial F}{\partial t} + c^2 \frac{\partial}{\partial x} (fE) &= -c(\mu + \sigma) F, \end{aligned} \tag{2.3}$$

where for the purposes of the present analysis we now consider the variable Eddington factor f to be known.

Recalling that B is given by (1.2), we find that the equations to be solved are (1.5) and (2.3) for the radiation energy density $E = E(\nu, x, t)$, the radiation flux $F = F(\nu, x, t)$, and the material temperature $\theta = \theta(x, t)$, subject to appropriate initial and boundary conditions.

Concerning the boundary conditions, one sees upon rewriting (2.3) in characteristic form that the following quantities U and V propagate (in time) spatially to the right and left, respectively, where

$$U = E + \frac{1}{c\sqrt{f}} F,$$

$$V = E - \frac{1}{c\sqrt{f}} F.$$

Hence the values of U may be prescribed at a left boundary while the values of V are given at a right boundary. In practice the boundary conditions are often specified in terms of E and F, and the required data for U and V (or their equivalent) are then retrieved from the given boundary conditions.

Initial conditions are given specifying the values of E, F, and θ (or equivalently U, V, and θ) at some fixed time, say $t = 0$.

3. Discretization into Frequency Groups

We partition the frequency interval $0 \leqslant \nu < \infty$ into J subintervals as

$$0 = \nu_0 < \nu_1 < \nu_2 < \cdots < \nu_{J-1} < \nu_J = \infty,$$

and rewrite Eq. (1.5) as

$$\epsilon_m'(\theta)\frac{\partial\theta}{\partial t} = -\sum_{j=1}^{J}\int_{\nu_{j-1}}^{\nu_j} c\mu\left[\theta^3\frac{4\pi}{c}B\left(\frac{\nu}{\theta},1\right) - E\right]d\nu. \tag{3.1}$$

Letting a denote the number

$$a = \frac{4\pi}{c}\int_0^\infty B(\sigma, 1)\,d\sigma, \tag{3.2}$$

and setting

$$\Phi = a\theta^4,$$

we then approximate (3.1) as

$$\frac{\partial\Phi}{\partial t} = -\kappa\sum_{j=1}^{J} c\mu_j[\Phi b_j - E_j], \tag{3.3}$$

where

$$\kappa = 4a\frac{\theta^3}{\epsilon_m'(\theta)},$$

and where μ_j denotes an average value of μ for ν on the j-th subinterval $\nu_{j-1} \leqslant \nu \leqslant \nu_j$,[3] while E_j denotes

$$E_j = E_j(x, t) = \int_{\nu_{j-1}}^{\nu_j} E(\nu, x, t)\,d\nu.$$

The numbers b_j $(j = 1, 2,..., J)$ are defined by

$$ab_j = \frac{4\pi}{c}\int_{\nu_{j-1}/\theta}^{\nu_j/\theta} B(\sigma, 1)\,d\sigma,$$

so that in particular there holds (see (3.2))

$$\sum_{j=1} b_j = 1. \tag{3.4}$$

[3] The Planck average is used in Ref. [1] for μ_j both in equation (3.3) and in the first equation of (3.5) below, while the Rosseland average is used in the last equation of (3.5) so as to obtain proper limiting behavior for the solution in the optically thin and diffusion cases. For simplicity, our analysis is based on using the same average for μ_j in all equations.

Finally we integrate both sides of the equations of (2.3) with respect to ν over the j-th frequency group $\nu_{j-1} \leqslant \nu \leqslant \nu_j$, and approximate the resulting equations as

$$\begin{aligned} &\frac{\partial E_j}{\partial t} + \frac{\partial F_j}{\partial x} = c\mu_j[\Phi b_j - E_j], \\ &\frac{\partial F_j}{\partial t} + c^2 \frac{\partial}{\partial x}(f_j E_j) = -c(\mu_j + \sigma_j) F_j, \qquad j = 1, 2, \ldots, J, \end{aligned} \tag{3.5}$$

where σ_j and f_j denote approximate values of σ and f for $\nu_{j-1} \leqslant \nu \leqslant \nu_j$ and F_j denotes

$$F_j = F_j(x, t) = \int_{\nu_{j-1}}^{\nu_j} F(\nu, x, t)\, d\nu.$$

Hence we have the approximate equations (3.3) and (3.5) which are to be solved for $\Phi = \Phi(x, t)$, $E_j = E_j(x, t)$, and $F_j = F_j(x, t)$ subject to appropriate initial and boundary conditions. These equations are further discretized with respect to x and t, and the resulting discrete equations are finally solved.

The coefficients μ_j, σ_j, b_j, f_j, and κ are treated explicitly in Ref. [1]; their values are computed as functions of x at a time level t and then held fixed while advancing the solution in time to level $t + \Delta t$, after which new values of the coefficients are computed. For purposes of the present analysis we shall assume that the coefficients μ_j, σ_j, and κ are known nonnegative functions of x independent of t, and we assume that b_j and f_j are known positive constants with $b_1, \ldots, b_J$ satisfying (3.4). This amounts to a certain linearization of the equations. We also introduce new flux variables $\mathfrak{F}_j$ by the formula

$$\mathfrak{F}_j = \frac{1}{c\sqrt{f_j}} F_j \qquad (j = 1, \ldots, J).$$

Then the equations (3.3) and (3.5) can be given as

$$\begin{aligned} &\frac{\partial E_j}{\partial t} + c\sqrt{f_j} \frac{\partial}{\partial x} \mathfrak{F}_j = c\mu_j(\Phi b_j - E_j), \\ &\frac{\partial \mathfrak{F}_j}{\partial t} + c\sqrt{f_j} \frac{\partial E_j}{\partial x} = -c(\mu_j + \sigma_j) \mathfrak{F}_j, \qquad j = 1, 2, \ldots, J, \\ &\frac{\partial \Phi}{\partial t} = -\kappa \sum_{j=1}^{J} c\mu_j(\Phi b_j - E_j). \end{aligned} \tag{3.6}$$

Finally, we find it convenient to choose the frequency partition $0 = \nu_0 < \nu_1 < \cdots < \nu_{J-1} < \nu_J = \infty$ so that the positive weights $b_1, \ldots, b_J$ are all equal (see (3.4)),

$$b_j = \frac{1}{J} \quad \text{for} \quad j = 1, \ldots, J. \tag{3.7}$$

This amounts to using a special Lagrangian coordinate system in frequency space.

4. The Difference Equations; Stability

We consider the initial value problem for the hyperbolic system (3.6). Specifically we seek functions $E_j = E_j(x, t)$, $\mathfrak{F}_j = \mathfrak{F}_j(x, t)$ $(j = 1, 2, \ldots, J)$, and $\Phi = \Phi(x, t)$ satisfying (3.6) in the region $-\infty < x < \infty$, $t \geqslant 0$ and subject to the usual initial conditions

$$\begin{aligned} &\Phi(x, 0), \quad E_j(x, 0), \quad \text{and} \quad \mathfrak{F}_j(x, 0) \quad \text{prescribed} \\ &\text{for} \quad j = 1, \ldots, J \quad \text{on} \quad -\infty < x < \infty \quad \text{at} \quad t = 0. \end{aligned} \tag{4.1}$$

Let the region $-\infty < x < \infty$, $t \geqslant 0$ be covered with a lattice of discrete points with coordinates (x_i, t_n) given as

$$x_i = i\Delta x, \qquad i = 0, \pm 1, \pm 2, \ldots,$$

$$t_n = n\Delta t, \qquad n = 0, 1, 2, \ldots,$$

where for simplicity we use constant mesh widths Δx and Δt throughout. Then setting $E_{ij}^n = E_j(x_i, t_n)$, $\mathfrak{F}_{ij}^n = \mathfrak{F}_j(x_i, t_n)$, $\Phi_i^n = \Phi(x_i, t_n)$, $\mu_{ij} = \mu_j(x_i)$, $\sigma_{ij} = \sigma_j(x_i)$, and $\kappa_i = \kappa(x_i)$, we consider the following difference analog of (3.6),

$$\begin{aligned} &\frac{E_{ij}^{n+1} - E_{ij}^n}{\Delta t} + c\sqrt{f_j}\,\frac{\mathfrak{F}_{i+1,j}^{n+1} - \mathfrak{F}_{ij}^{n+1}}{\Delta x} = c\mu_{ij}(\Phi_i^{n+1} b_j - E_{ij}^{n+1}), \\ &\frac{\mathfrak{F}_{ij}^{n+1} - \mathfrak{F}_{ij}^n}{\Delta t} + c\sqrt{f_j}\,\frac{E_{ij}^{n+1} - E_{i-1,j}^{n+1}}{\Delta x} = -c(\mu_{ij} + \sigma_{ij})\,\mathfrak{F}_{ij}^{n+1}, \\ &\frac{\Phi_i^{n+1} - \Phi_i^n}{\Delta t} = -\kappa_i \sum_{j=1}^{J} c\mu_{ij}(\Phi_i^{n+1} b_j - E_{ij}^n). \end{aligned} \tag{4.2}$$

The forward and backward spatial differences appearing respectively in the first and second equations of (4.2) are used in order to take into account a symmetric portion of the domain of dependence of the system (3.6), and lead to improved

accuracy for the difference scheme as compared with similar noncharacteristic first-order accurate schemes with other nonsymmetric forms for the spatial differences (cf. Ref. [13]).

The equations (4.2) are partially explicit due to the term E_{ij}^n appearing in the sum in the last equation. This formulation has been deemed necessary (rather than a fully implicit scheme which should be more stable) so as to make the resulting linear algebraic equations simple (cheap) to solve for the initial-boundary value problem on a bounded interval. In fact we can use the last two equations of (4.2) to eliminate $\mathfrak{F}^{n+1}$ and Φ^{n+1} in the first equation of (4.2), and the resulting system of simultaneous linear equations for E^{n+1} can then be simply solved since it has a coefficient matrix of tridiagonal form (cf. Ref. [1]).

Along with (4.2) we have discretized initial values gotten similarly from (4.1),

$$\Phi_i^0,\ E_{ij}^0,\ \mathfrak{F}_{ij}^0 \qquad \text{prescribed for} \quad i = 0, \pm 1, \pm 2,\ldots, \quad j = 1, 2,\ldots, J \tag{4.3}$$

We shall use a vector X^n to denote any solution of (4.2) evaluated at time $t_n = n\Delta t$, where

$$X^n = \{E_{ij}^n,\ \mathfrak{F}_{ij}^n,\ \Phi_i^n \quad j = 1,\ldots, J;\ i = 0, \pm 1,\ldots\}. \tag{4.4}$$

Also we shall use the following energy (l_2) norm $\| \cdot \|$ defined by the formula

$$\| X^n \|^2 = \sum_{j=1}^{J} \sum_{i=-\infty}^{\infty} \{\kappa_i[(1 + c\mu_{ij}\,\Delta t)(E_{ij}^n)^2 + (\mathfrak{F}_{ij}^n)^2] + (\Phi_i^n\, b_j)^2\}. \tag{4.5}$$

We consider initial data (4.3) for which $\| X^0 \| < \infty$, and then study "physically reasonable" solutions of (4.2) for which $\| X^n \|$ remains finite for each fixed n,

$$\| X^n \| \leqslant K_n, \tag{4.6}$$

for some suitable constant K_n depending in general on the particular solution considered and on the time level n. In fact on physical grounds we need only consider solutions for which the energies and fluxes decay to zero as $x = i\Delta x \to \pm\infty$, which amounts to the condition (4.6).

We shall say that the difference scheme (4.2) is *unconditionally stable* if there holds an estimate of the form

$$\| X^n \| \leqslant C \cdot \| X^0 \|$$

for some fixed constant C independent of n and for all solutions satisfying (4.6) and for all $\Delta t > 0$.[4] In this case the numbers $\| X^n \|$ (which are finite for each fixed n) cannot grow unduly with *increasing n*.

[4] The constant C is independent of Δt and independent of the particular solution considered.

THEOREM. *The difference scheme* (4.2) *is unconditionally stable. In fact,*

$$\| X^n \| \leqslant \| X^0 \| \tag{4.7}$$

for all n and any $\Delta t > 0$.

Proof. We multiply the first equation of (4.2) by $\kappa_i E_{ij}^{n+1}$, the second equation by $\kappa_i \mathfrak{F}_{ij}^{n+1}$, and the last equation by $\Phi_i^{n+1} b_j^2$. Upon adding the resulting expressions and summing over j we find after some manipulation the result

$$\begin{aligned}
&\sum_{j=1}^{J} \Big\{ \kappa_i (1 + c\,\Delta t\,\mu_{ij})(E_{ij}^{n+1})^2 + \kappa_i [1 + c\,\Delta t(\mu_{ij} + \sigma_{ij})](\mathfrak{F}_{ij}^{n+1})^2 \\
&\qquad + [1 + \kappa_i c\,\Delta t(\mu_i \cdot b)](\Phi_i^{n+1} b_j)^2 \\
&\qquad + \frac{\kappa_i c \sqrt{f_j}\,\Delta t}{\Delta x} (E_{ij}^{n+1} \mathfrak{F}_{i+1,j}^{n+1} - E_{i-1,j}^{n+1} \mathfrak{F}_{ij}^{n+1}) \Big\} \\
&= \sum_{j=1}^{J} \{ \kappa_i (E_{ij}^n E_{ij}^{n+1} + \mathfrak{F}_{ij}^n \mathfrak{F}_{ij}^{n+1}) + (\Phi_i^n b_j)(\Phi_i^{n+1} b_j) \\
&\qquad + \kappa_i c\,\Delta t [\mu_{ij} E_{ij}^{n+1} + (\mu_i \cdot E_i^n)\, b_j]\, \Phi_i^{n+1} b_j \},
\end{aligned} \tag{4.8}$$

where

$$\mu_i \cdot b = \sum_{j=1}^{J} \mu_{ij} b_j , \qquad \mu_i \cdot E_i^n = \sum_{j=1}^{J} \mu_{ij} E_{ij}^n .$$

Using the simple inequality $ab \leqslant (a^2 + b^2)/2$ which holds for all real numbers a and b, we find easily the estimate

$$\begin{aligned}
&\sum_{j=1}^{J} \{ \kappa_i (E_{ij}^n E_{ij}^{n+1} + \mathfrak{F}_{ij}^n \mathfrak{F}_{ij}^{n+1}) + (\Phi_i^n b_j)(\Phi_i^{n+1} b_j) \} \\
&\leqslant \tfrac{1}{2} \sum_{j=1}^{J} \{ \kappa_i [(E_{ij}^n)^2 + (\mathfrak{F}_{ij}^n)^2] + (\Phi_i^n b_j)^2 \} \\
&\qquad + \tfrac{1}{2} \sum_{j=1}^{J} \{ \kappa_i [(E_{ij}^{n+1})^2 + (\mathfrak{F}_{ij}^{n+1})^2] + (\Phi_i^{n+1} b_j)^2 \}.
\end{aligned} \tag{4.9}$$

Similarly we find

$$\sum_{j=1}^{J} \kappa_i c\, \Delta t\, \mu_{ij} E_{ij}^{n+1} \Phi_i^{n+1} b_j \leqslant \tfrac{1}{2} \sum_{j=1}^{J} \kappa_i c\, \Delta t\, \mu_{ij} \{(E_{ij}^{n+1})^2 + (\Phi_i^{n+1} b_j)^2\}$$

$$= \tfrac{1}{2} \sum_{j=1}^{J} \kappa_i c\, \Delta t [\mu_{ij} (E_{ij}^{n+1})^2 + (\mu_i \cdot b)(\Phi_i^{n+1} b_j)^2], \tag{4.10}$$

where we used (3.7) and (3.4) to rewrite the term involving Φ_i^{n+1}. Finally we consider the last term on the right side of (4.8), where

$$\sum_{j=1}^{J} \kappa_i c\, \Delta t (\mu_i \cdot E_i^n)\, b_j \Phi_i^{n+1} b_j = \sum_{j=1}^{J} \sum_{j'=1}^{J} \kappa_i c\, \Delta t\, \mu_{ij'} E_{ij'}^n b_j \Phi_i^{n+1} b_j .$$

Since $b_j = b_{j'}$ by (3.7), we find upon interchanging the order of the j and j' summations the result

$$\sum_{j=1}^{J} \kappa_i c\, \Delta t (\mu_i \cdot E_i^n)\, b_j \Phi_i^{n+1} b_j = \sum_{j=1}^{J} \kappa_i c\, \Delta t\, \mu_{ij} E_{ij}^n \Phi_i^{n+1} b_j , \tag{4.11}$$

which can be estimated as in (4.10) with E_{ij}^n replacing E_{ij}^{n+1} there.

Using now the inequalities (4.9) and (4.10) along with the similar inequality obtained from (4.11), we find from (4.8), after some cancellation, the result

$$\tfrac{1}{2} \sum_{j=1}^{J} \Big\{ \kappa_i (1 + c\, \Delta t\, \mu_{ij})(E_{ij}^{n+1})^2 + \kappa_i [1 + 2c\, \Delta t (\mu_{ij} + \sigma_{ij})](\mathfrak{F}_{ij}^{n+1})^2$$

$$+ (\Phi_i^{n+1} b_j)^2 + \frac{2\kappa_i c \sqrt{f_j}\, \Delta t}{\Delta x} (E_{ij}^{n+1} \mathfrak{F}_{i+1,j}^{n+1} - E_{i-1,j}^{n+1} \mathfrak{F}_{ij}^{n+1}) \Big\}$$

$$\leqslant \tfrac{1}{2} \sum_{j=1}^{J} \{ \kappa_i [(1 + c\, \Delta t\, \mu_{ij})(E_{ij}^n)^2 + (\mathfrak{F}_{ij}^n)^2] + (\Phi_i^n b_j)^2 \}.$$

Summing this inequality over all i and using (4.5) we find the estimate

$$\tfrac{1}{2} \| X^{n+1} \|^2 + \sum_{j=1}^{J} \sum_{i=-\infty}^{\infty} \kappa_i c\, \Delta t (\mu_{ij} + \sigma_{ij})(\mathfrak{F}_{ij}^{n+1})^2 \leqslant \tfrac{1}{2} \| X^n \|^2, \tag{4.12}$$

where we used also the result

$$\sum_{i=-\infty}^{\infty} \frac{\kappa_i c \sqrt{f_j}\, \Delta t}{\Delta x} (E_{ij}^{n+1} \mathfrak{F}_{i+1,j}^{n+1} - E_{i-1,j}^{n+1} \mathfrak{F}_{ij}^{n+1}) = 0,$$

which is easily seen to hold for any solution satisfying (4.6). The stated result (4.7) now follows directly from (4.12), completing the proof of the theorem.

Even though our results refer here to the pure initial value problem, numerical tests have indicated that the difference scheme (4.2) is unconditionally stable also for the mixed initial-boundary value problem on a bounded interval (cf. Ref. [1]).

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REFERENCES

1. B. E. FREEMAN, ———— "The VERA System of Radiation-Hydrodynamics Codes," Report DASA 2258-1, Defense Atomic Support Agency, 1969.
2. G. C. POMRANING AND B. E. FREEMAN, The equation of radiative transfer with scattering, *J. Quant. Spectrosc. Radiat. Transfer* **8** (1968), 909–917.
3. B. WENDROFF, A difference scheme for radiative transfer, *J. Comput. Phys.* **4** (1969), 221–229.
4. P. M. CAMPBELL, "Finite Difference Methods for Nonlinear Radiation Transport," Report UCRL-12411, Lawrence Radiation Laboratory, Calif., 1965.
5. R. D. RICHTMYER AND K. MORTON, "Difference Methods for Initial Value Problems," 2nd Edition, Wiley (Interscience), New York, 1967.
6. H. B. KELLER AND B. WENDROFF, On the formulation and analysis of numerical methods for time Dependent transport equations, *Comm. Pure Appl. Math.* **10** (1957), 567–582.
7. B. WENDROFF, "An Unconditionally Stable Finite Difference Method for Photon Transport Problems," Report LAMS-2199, Los Alamos Scientific Laboratory, N.Y., 1958.
8. B. WENDROFF, The structure of certain finite difference schemes, *SIAM Rev.* **3** (1961), 237–242.
9. R. P. PEARCE AND A. R. MITCHELL, On finite difference methods of solution of the transport equation, *Math. Comp.* **16** (1962), 155–169.
10. A. S. EDDINGTON, On the radiative equilibrium of the stars. *Mon. Notic. Roy. Astron. Soc.* **77**, No. 16, (1916), 16–35.
11. G. SPILLMAN, "Test Calculations with Variable Eddington Factor Radiation Approximation," Internal Los Alamos Scientific Laboratory Memorandum, June 28, 1969.
12. G. C. POMRANING, An extension of the Eddington approximation, *J. Quant. Spectrosc. Radiat. Transfer* **9** (1969), 407–422.
13. F. JOHN, "Lectures on Advanced Numerical Analysis," Gordon and Breach, New York, 1967.
14. J. B. ANDREWS, II, AND K. F. HANSEN, Numerical solution of the time-dependent multigroup diffusion equations, *Nucl. Sci. Eng.* **31** (1968), 304–313.